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DESIGN OF BLOCK-BACKSTEPPING CONTROLLER TO BALL AND ARC SYSTEM BASED ON ZERO DYNAMIC THEORY

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Abstract

This paper develops a proposed block-backstepping algorithm for balancing and tracking control of ball and arc system. Two block-backstepping designs have been presented; one from the linearized model and other from a nonlinear model of the considered underactuated system. Also, two main control objectives have been achieved; firstly to bring the ball to rest on the top of the arc and secondly to make the cart track a defined reference trajectory. Moreover, integral action is included in the developed block-backstepping control law to improve the steady-state characteristics and to enhance the robustness of the overall system. Additionally, the internal stability of the nonlinear system has been analyzed using zero dynamic criteria to guarantee the global asymptotic stability at the desired equilibrium point. The performance of the designed control algorithm is assessed via simulated results. The results show that the block-backstepping controller designed for nonlinear system gives better transient performance than that designed for the linear system. Also, the nonlinear controller can cope with larger initial angular ball position without loss of stability.

Keywords: Ball and arc system, Block backstepping, Lyapunov, Tracking control.

1. Introduction

Underactuated Mechanical Systems (UMSs) are characterized by having fewer actuating inputs than controlled variables. Such systems can be shown in many applications like underwater vehicles, aircraft, mobile robot, inverted pendulum systems, helicopter, space robot and underactuated manipulator [1-3].

The ball and arc system is one of UMS with two DOF that has been proposed for demonstrating the basic concepts of modern control theory [4]. This system can be described by a ball that rolls on a top of an arc. The arc sits on a cart driven by a motor as depicted in Fig. 1.

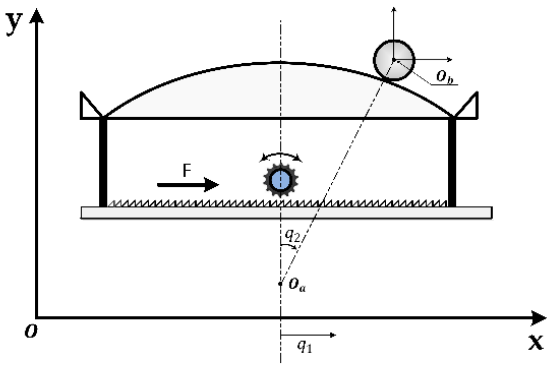


Fig. 1. Schematic representation of ball and arc system.

The controller task of such system is to balance the ball on the boundary of the arc and to position the carriage, ball and arc assembly, at the midpoint of the track through an actuating motor [5]. Several approaches of control system were presented in the literature to control the ball and arc system, such as optimal control [6], T-S Fuzzy Model [7], optimal and disturbance-accommodating control [8], and sliding mode control [5].

During the period 1987-1989, the idea of integrator backstepping was proposed and developed by Koditschek [9], Sonntag and Sussmann [10], Tsinias [11], Byrnes and Isidori [12]. In 1989, Sontag and Sussmann had established the stabilization basis of backstepping via an integrator. Krstic et al. described in detail the adaptive and nonlinear Backstepping designs in 1995 [13]. The structure of backstepping compromises methods for parameter adaptation, tuning functions, and modular designs for both full state feedback and output feedback (observer backstepping).

Recently, several researchers have attempted to reach more generalized backstepping algorithms that can successfully deal with the stabilization problems of complicated nonlinear systems. Block backstepping method is one of the most productive backstepping based algorithm. This control strategy can address the control problem of various nonlinear (MIMO) systems [14-19]. For the plant dynamic equations to be controlled through block-backstepping design, two conditions have to be satisfied [13]:

- The first step of block-backstepping design, the dynamic equations are transformed into a block strict-feedback form.

- The second step; backstepping procedure may be applied to each state block to derive the expression of control input for the overall nonlinear system.

The salient feature of block backstepping control strategy is that it can be applied to a class of systems whose dynamic equations are not in strict-feedback form, and it may also improve the problem of ‘explosion of complexity’ [18].

The motivation behind the present work is that the underactuated model of ball and arc system is not in a strict-feedback form and, also, it is characterized by high complexity. Therefore, backstepping design and control of the considered systems is a challenging problem, whose solution is the motivation of the work.

This contribution of the work can be summarized by the following points:

- A novel block-backstepping design is applied to solve the control problem of the ball and arc system.
- The control problem is considered for two cases (linearized and nonlinear systems), where control structures are developed, derived and analyzed. In the case of the linearized model, the block-backstepping is designed to achieve the control objectives within the stabilization zone such that all states are ensured to converge to a defined trajectory. Then, another block-backstepping design is presented to a nonlinear system based on the information from the design of the linearized system.
- Lyapunov stability theorem is used to analyze the asymptotic stability of the overall system, while the internal stability of the dynamic equations is analyzed using zero dynamic criteria to achieve GAS at its desired equilibrium point.
- Finally, integral action is included to improve the steady state performance of the controller.

This is organized as follows; section two presents the modelling of ball and arc system. A novel block Backstepping control algorithm for the linearized system is developed in section three, while a novel development of block backstepping controller for the nonlinear system is given in section four. Zero dynamics and stability analysis are included in section five and six, respectively. The simulated results are presented in section seven. In section eight, conclusions based on simulated results have been drawn.

2.Mathematical Model of Ball and Arc System

2.1. Nonlinear model

In this sub-section, the mathematical model of the ball and arc system is set up using the Euler-Lagrange formulation [5].

$$\left(M+m\right)\ddot{q}_1+m\left(R+r\right)\left[\ddot{q}_2\cos q_2-\dot{q}_2^2\sin q_2\right]=F \tag{1}$$

$$m\left(R+r\right)\ddot{q}_1\cos q_2+(m(R+r)^2+I(\frac{R+r}{r})^2)\ddot{q}_2-mg\left(R+r\right)\sin q_2=0 \tag{2}$$

$$F=\frac{k_m}{R_aR_1}u-\frac{k_m^2}{R_aR_1^2}\dot{q}_1 \tag{3}$$

where, F is the mechanical force applied to the cart, and u is the control input (in volt) of the ball and arc system. In order to keep the ball on the arc, the centripetal force is assumed to be high such that the following condition has to be satisfied,

$$N = mg\cos q_2 - m\left(R+r\right)\dot{q}_2^2 > 0 \tag{4}$$

where N is the normal reaction force due to the arc. The states of the system is described by the vector [q_1 p_1 q_2 p_2], where q_1 is the displacement of the cart mass center, p_1 is the velocity of the cart, q_2 is the angular displacement between the vertical and the line through the center of the ball O_b and the center of the arc O_a, and p_2 is the angular velocity of the ball. Then, the above equations can formulate as follows:

$$\begin{aligned}
\dot{q}_1 &= p_1 \\
\dot{p}_1 &= \frac{1}{\beta(q)}\left(m_{12}m_{22}p_2^2\sin q_2 - m_{12}^2 g\sin q_2\cos q_2 + m_{22}\left(\frac{k_m}{R_aR_1}u - \frac{k_m^2}{R_aR_1^2}p_1\right)\right) \\
\dot{q}_2 &= p_2 \\
\dot{p}_2 &= \frac{1}{\beta(q)}\left(m_{11}m_{12}\, g\sin q_2 - m_{12}^2 p_2^2\sin q_2\cos q_2 - m_{12}\cos q_2\left(\frac{k_m}{R_aR_1} - \frac{k_m^2}{R_aR_1^2}p_1\right)\right)
\end{aligned} \tag{5}$$

where,

$$M = \begin{bmatrix} m_{11} & m_{12}\cos q_2 \\ m_{21}\cos q_2 & m_{22} \end{bmatrix},\ m_{11} = M+m,\ \ m_{12} = m\left(R+r\right) = m_{21}$$

$$m_{22} = m\left(R+r\right)^2 + I\left(\frac{R+r}{r}\right)^2,$$

$$\beta(q) = m_{11}I\left(\frac{R+r}{r}\right)^2 + Mm\left(R+r\right)^2 + m_{12}^2\sin^2 q_2 > 0$$

It is noteworthy to mention that the control objectives are not only to maintain the ball to stable on the top of the arc, but also the cart achieves the trajectory tracking of the defined reference trajectory.

2.2. Linearized model

The nonlinear system in Eq. (5) was linearized nearby the equilibrium point $\left(q,\ p\right) = 0$. In order to realize and analyze the properties of the ball-arc system, the disturbance was ignored for simplicity [20].

$$\begin{aligned}
\dot{q}_1 &= p_1 \\
\dot{p}_1 &= \frac{1}{h}\left(-m_{12}^2\, gq_2 - \frac{m_{22}k_m^2}{R_aR_1^2}p_1 + \frac{m_{22}k_m}{R_aR_1}u\right) \\
\dot{q}_2 &= p_2 \\
\dot{p}_2 &= \frac{1}{h}\left(m_{11}m_{12}gq_2 + \frac{m_{12}k_m^2}{R_aR_1^2}p_1 - \frac{m_{12}k_m}{R_aR_1}u\right)
\end{aligned} \tag{6}$$

where,

$$m_{11} = M + m\,,\; m_{22} = m\left(R+r\right)^2 + I\left(\frac{R+r}{r}\right)^2,\; m_{12} = m\left(R+r\right) = m_{21}\,,$$

$$h = m_{11} I\left(\frac{R+r}{r}\right)^2 + M\;\, m\left(R+r\right)^2$$

3. Control Design Algorithm for Linearized Model

The block backstepping control algorithm for the linearized version of the system is proposed to achieve the control objectives within a stabilization zone in the neighbour of equilibrium point [21]. The block diagram of the closed-loop system is shown in Fig. 2.

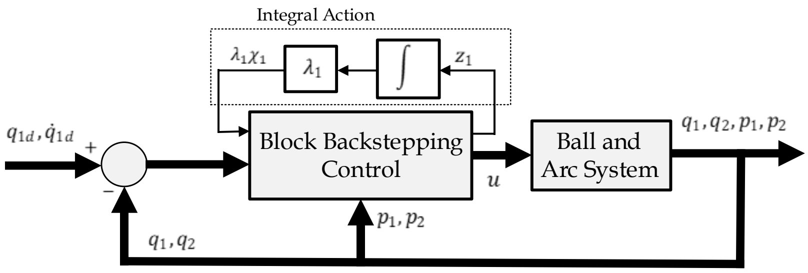


Fig. 2. The block diagram of block backstepping based control for ball and arc system.

The next steps include the design procedure for the application of block-backstepping control to Linearized model:

Step 1: The regulated variable is first introduced as

$$z_1 = q_2 + k_1 e + k_2\left(m_{12}\,\dot{e} + m_{22}\; p_2\right) \tag{7}$$

$$e = q_1 - q_{1d} \tag{8}$$

$$\dot{e} = p_1 - \dot{q}_{1d} \tag{9}$$

where, k_1 and k_2 are design constants. Taking the derivative of z_1 to have:

$$\dot{z}_1 = \dot{q}_2 + k_1\dot{e} + k_2\left(m_{12}\ddot{e} + m_{22}\dot{p}_2\right) \tag{10}$$

or,

$$\dot{z}_1 = p_2 + k_1\dot{e} + k_2 m_{12} g\, q_2 - k_2 m_{12}\,\ddot{q}_{1d} \tag{11}$$

The state variable p_2 is taken as a virtual control variable, for which the following stabilizing function is chosen

$$\alpha = -k_1\dot{e} - c_1 z_1 - \lambda_1\chi_1 - k_2 m_{12} g q_2 + k_2 m_{12}\,\ddot{q}_{1d} \tag{12}$$

where c_1 is a positive design constant and λ is a real-valued design constant. The integral action of the regulated variable is incorporated with the controller to guarantee the convergence of the regulated variable to zero at steady state in the presence of the disturbances and inaccuracy of the system.

$$\chi_1 = \int_0^t z_1 dt \tag{13}$$

The corresponding error variable is defined as

$$z_2 = p_2 - \alpha \tag{14}$$

Consequently, the time derivative of z_1 is expressed as following

$$\dot{z}_1 = z_2 - c_1 z_1 - \lambda_1 \chi_1 \tag{15}$$

Step 2: The derivative of $\boldsymbol{z}_2$ is computed as follows:

$$\dot{z}_2 = \dot{p}_2 - \dot{\alpha} \tag{16}$$

$$\dot{\alpha} = -k_1 \ddot{e} - c_1 \dot{z}_1 - \lambda_1 z_1 - k_2\, m_{12}\, g\, \dot{q}_2 + k_2 m_{12}\, \ddot{q}_{1d} \tag{17}$$

One can show that Eq. (16) can be given by

$$\dot{z}_2 = \psi\, u + \lambda_1 z_1 + c_1 \left(z_2 - c_1 z_1 - \lambda_1 \chi_1 \right) + \phi \tag{18}$$

where, ψ and ϕ is given by

$$\psi = \frac{k_m}{R_1 R_a h} \left(k_1 m_{22} - m_{12} \right)$$

$$\phi = \frac{1}{h}\left(g\ m_{11} m_{12} q_2 + \frac{k_m^2\ m_{12}\, p_1}{R_1^2 R_a}\right) - \frac{k_1}{h}\left(\ddot{q}_{1d}\ h + g\ m_{12}^2 q_2 + \frac{k_m^2 m_{22}\, p_1}{R_1^2\ R_a}\right) + g k_2 m_{12} p_2 - \ddot{q}_{1d}\ k_2 m_{12}$$

The desired dynamics of z_2 is expressed as follows:

$$\dot{z}_2 = -z_1 - c_2 z_2 \tag{19}$$

Substituting Eq. (19) into Eq. (18) and solving for the control signal to achieve the desired dynamics of the z_1 and z_2 the linear controller is obtained:

$$u = \psi^{-1}\left(-(1 - c_1^2 + \lambda_1) z_1 - (c_1 + c_2) z_2 + \lambda_1 c_1 \chi_1 - \phi\right) \tag{20}$$

where ψ is invertible, c_2 is a positive design constant. The stability of the system is analyzed based on the following Lyapunov function;

$$V = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{2} \lambda_1\, \chi_1^2 \tag{21}$$

From Eq. (11) to Eq. (20), the time derivative of Eq. (21) is determined as follows:

$$\dot{V} = z_1 \dot{z}_1 + z_2 \dot{z}_2 + \lambda_1 \chi_1 \dot{\chi}_1 \tag{22}$$

$$= z_1 \left(z_2 - c_1 z_1 - \lambda_1 \chi_1 \right) + \lambda_1 z_1 \chi_1 + z_2 \left(\psi u + \lambda_1 z_1 + c_1 \left(z_2 - c_1 z_1 - \lambda_1 \chi_1 \right) + \phi \right)$$

Then, the following inequality can be concluded

$$\dot{V} = -c_1 z_1^2 - c_2 z_2^2 \leq 0 \tag{23}$$

It is evident from the above equation that the inequality $V(t) \le V(0)$ is verified and, hence, the states χ_1, z_1 and z_2 are bounded and consequently $\dot{z}_1$, $\dot{z}_2$ are also bounded. The second derivative of a Lyapunov function can easily be computed as:

$$\ddot{V} = -2c_1\, z_1\, \dot{z}_1 - 2c_2\, z_2\, \dot{z}_2 \tag{24}$$

Since z_1, z_2, $\dot{z}_1$ and $\dot{z}_2$ are all bounded, therefore, $\ddot{V}$ is also bounded. Barbalat's Lemma can be applied to show that both z_1 and z_2 converge to zero as $t \to \infty$. The zero dynamic of the system is computed as follows:

$$\begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} \tag{25}$$

where,

$$a_1 = \frac{m_{12} g k_1}{\left(k_1 m_{22} - m_{12}\right)},\ a_2 = \frac{m_{12}^2 k_2 g}{\left(k_1 m_{22} - m_{12}\right)},\ \gamma = \frac{1}{\left(k_1 m_{22} - m_{12}\right)} \left(m_{12}\left(\ddot{q}_{1d} + k_2 m_{22}\, \ddot{q}_{1d}\right)\right)$$

The matrix is a Hurwitz matrix if a_1 and a_2 are less than zero. So that k_1 and k_2 are chosen to satisfy the following inequalities:

$$k_2 > 0, \qquad 0 < k_1 < \frac{m_{12}}{m_{22}} \tag{26}$$

Then the states q_1 and p_1 will converge to zero. Hence, the proposed control law guarantees the global stabilization of the ball and arc system.

4. Control Design Algorithm for Non-Linear Model

In a previous analysis, Lyapunov stability analysis has been applied to synthesize a block-backstepping controller for the linearized system described by Eq. (6). However, if the same controller in Eq. (20) is applied to the original system in Eq. (5) then it is expected to work well only nearby the equilibrium points within the stabilization zone. Therefore, it is necessary to design a novel block backstepping controller for a nonlinear system, which can cope with system complexity and can bring the ball to rest on the top of the arc starting outside the stabilization zone. In this complex nonlinear model, the key point is to choose a suitable initial nonlinear regulated variable. Thus, the regulated variable in the previous procedure is considered, here again; then the regulated variable is continuously updated until finding an appropriate variable. Therefore, the design concept used for the linearized system is extended to design the complete nonlinear system. The regulation variable is modified from the physical and math structure for designing nonlinear controller. The design procedure of application of blockstepping control design to nonlinear model can be is summarized as follows:

Step 1: The variable to be regulated is chosen as:

$$z_3 = q_2 + k_3 e + k_4\left(m_{12}\cos q_2 \dot{e} + m_{22} p_2\right) \tag{27}$$

$$e = q_1 - q_{1d} \tag{28}$$

$$\dot{e} = p_1 - \dot{q}_{1d} \tag{29}$$

where k_3 and k_4 are design constants. Then, one can obtain the time derivative of z_3 as:

$$\dot{z}_3 = \dot{q}_2 + k_3\dot{e} + k_4\left(-m_{12}\dot{q}_2 \sin q_2 \dot{e} + m_{12} \cos q_2 \ddot{e} + m_{22}\dot{p}_2\right)$$

The above equation can be given as,

$$\dot{z}_3 = p_2 + k_3\dot{e} + k_4(-m_{12}\,\dot{e}p_2 \sin q_2 - m_{12}\cos q_2\,\ddot{q}_{1d} - \frac{0.5m_{12}^3\,g\sin 2q_2\cos q_2}{\beta(q)} + \frac{m_{11}\;m_{12}\,m_{22}\,g\sin q_2}{\beta(q)}) \tag{30}$$

The following stabilizing function has been chosen in the design procedure;

$$\alpha = -\left(k_3 - k_4 m_{12} p_2 \sin q_2\right)\dot{e} + k_4 m_{12}\cos q_2\,\ddot{q}_{1d} - c_3 z_3 - \lambda_2\chi_1 + \frac{0.5g\,k_4\,m_{12}^3\sin 2q_2\cos q_2}{\beta(q)} - \frac{g\,k_4 m_{11} m_{12} m_{22}\sin q_2}{\beta(q)} \tag{31}$$

where c_3 a positive design is constant, λ_2 is a design constant. The integral action of the regulated variable is defined as:

$$\chi_1 = \int_0^t z_3\,dt \tag{32}$$

The corresponding error variable is defined as

$$z_4 = p_2 - \alpha \tag{33}$$

Consequently, the time derivative of z_3 is expressed as follows:

$$\dot{z}_3 = z_4 - c_3\,z_3 - \lambda_2\,\chi_1 \tag{34}$$

Step 2: The time derivative of z_4 is computed as follows:

$$\dot{z}_4 = \dot{p}_2 - \dot{\alpha} \tag{35}$$

Using Eqs. (5) and (35) one can show, after long calculation, that

$$\dot{z}_4 = \psi u + \lambda_2 z_3 + c_3\left(z_4 - c_3 z_3 - \lambda_2\chi_1\right) + \phi \tag{36a}$$

where ψ and ϕ are given by:

$$\psi = \frac{k_m}{\beta(q) R_1 R_a}\left(\left(k_3 - k_4 m_{12} p_2 \sin q_2\right)m_{22} - \left(1 - k_4 m_{12}\dot{e}\sin q_2\right)m_{12}\cos q_2\right) \tag{36b}$$

$$\phi = \frac{(\gamma_2+\gamma_3)\dot{\beta}(q) - (\dot{\gamma}_2+\dot{\gamma}_3)\beta(q)}{\beta(q)^2} - k_4\,m_{12}\dot{e}p_2^2\cos q_2 + \frac{\left(k_3 - k_4 m_{12} p_2 \sin q_2\right)}{\beta(q)} \left(-\beta(q)\ddot{q}_{1d} - g\cos q_2 \sin q_2 m_{12}^2 + m_{22}\sin q_2 m_{12}\,p_2^2 - \frac{k_m^2\,m_{22}\,p_1}{R_1^2\,R_a}\right) + \frac{\left(1-\dot{e}\;k_4\,m_{12}\sin q_1\right)}{\beta(q)}\left(g\,m_{11}m_{12}\sin q_2 - m_{12}^2\,p_2^2\cos q_2\sin q_2 + \frac{k_m^2\,m_{12}\,p_1\cos q_2}{R_1^2\,R_a}\right) + \ddot{q}_{1d}\,k_4\,m_{12}\,p_2\sin q_2 - \ddot{q}_{1d}\,k_4 m_{12}\cos q_2 \tag{36c}$$

$$\begin{aligned}
&\gamma_2 = -k_4 m_{11} m_{12} m_{22} g \sin q_2 \\
&\gamma_3 = 0.5 k_4 m_{12}^3 g \sin 2q_2 \cos q_2 \\
&\dot{\gamma}_2 = -k_4 m_{11} m_{12} m_{22} g p_2 \cos q_2 \\
&\dot{\gamma}_3 = -0.5 k_4 m_{12}^3 g p_2 \sin 2q_2 \sin q_2 + k_4 m_{12}^3 g p_2 \cos 2q_2 \cos q_2 \\
&\dot{\beta}(q) = m_{12}^2 \sin 2q_2
\end{aligned} \tag{36d}$$

The desired dynamics of z_4 can be given by the following expression

$$\dot{z}_4 = -z_3 - c_4 z_4 \tag{37}$$

Substituting Eq. (37) into Eq. (36) and solving for a nonlinear controller, which can achieve the desired dynamics of, z_3 and z_4 to have,

$$u = \psi^{-1}\left(-\left(1 - c_3^2 + \lambda_2\right) z_3 - \left(c_3 + c_4\right) z_4 + \lambda_2 c_3 \chi_1 - \phi\right) \tag{38}$$

It is clear from nonlinear controller described by Eq. (38) is more complex than that obtained for linearized system indicated by Eq. (20); the nonlinear controller contains nonlinear terms represented by cos q_2 and sin q_2, while the controller based on the linearized model is simpler and free from nonlinear terms.

5. Zero Dynamics Analysis

It is worthy to mention that the control input given by Eq. (38) does guarantee the stability of transformed variables z_2 and z_4. Meanwhile, algebraic state transformation defined by Eqs. (27) and (34) transforms the nonlinear dynamic equations of the plant into a reduced order state model described by z_2 and z_4.. In other word, the state transformation results in a second-order internal dynamics [22].

If the variable z_3 is considered as the system output and it is differentiated twice, then the following will result:

$$\begin{aligned}
&\ddot{z}_3 = \dot{z}_4 - c_3\dot{z}_3 - \lambda_2 z_3 = \psi u + \lambda_2 z_3 + c_3 \dot{z}_3 + \phi - c_1 \dot{z}_3 - \lambda_2 z_3 \\
&\ddot{z}_3 = \psi u + \phi
\end{aligned} \tag{39}$$

The setting z_3 = 0, then from Eq. (27) one can get

$$z_3 = q_2 + k_3 e + k_4\left(m_{12} \cos q_2 \dot{e} + m_{22} p_2\right) = 0$$

or

$$q_2 = -k_3 e - k_4\left(m_{12} \cos q_2 \dot{e} + m_{22} p_2\right) \tag{40}$$

Since z_3 = 0 it is logical that $\dot{z}_3 = 0$ and the following expression can be obtained based on Eq. (30)

$$p_2 = -k_3 \dot{e} - k_4\left(-m_{12} \dot{e} p_2 \sin q_2 - m_{12} \cos q_2 \ddot{q}_{1d} - \frac{0.5 g m_{12}^3 \sin 2q_2 \cos q_2}{\beta(q)} + \frac{m_{11} m_{12} m_{22} g \sin q_2}{\beta(q)}\right) \tag{41}$$

Also, $\ddot{z}_3 = 0$ to have,

$$\ddot{z}_3 = \psi u + \phi = 0$$

or,

$$u = -\psi^{-1}\phi \tag{42}$$

Therefore, one can represent the dynamics of q_1 and p_1 subsystem together with the input u in Eq. (42) as,

$$\begin{aligned} \dot{q}_1 &= p_1 \\ \dot{p}_1 &= f_1 + g_1 u \to f_1 - g_1\left(\psi^{-1}\phi\right) = F\left(q_1, p_1\right)\big|z_3 = 0 \end{aligned} \tag{43}$$

It is evident from the expressions of , ϕ , ψ , q_2 and p_2 in Eqs. (36b), (36c), (40) and (41) that it depends on the choice of the suitable parameter k , for this reason after substitute Eqs. (36b), (36c), (40) and (41) the zero dynamics in Eq. (43) solely depend on the parameter k to ensure desired characteristics of internal stability.

6. Stability Analysis

Lemma 1: For the nonlinear ball and arc system, the control input in Eq. (38) can perform a trajectory tracking of the defined reference trajectory. In particular, for any initial conditions $[q_1(0)\ p_1(0)\ q_2(0)\ p_2(0)]$, the trajectory tracking errors $[e(t)\ \dot{e}(t)\ \ q_2(t)\ \ p_2(t)]$ guarantees GAS as $t \to \infty$ under the operation of the control input law expressed in Eq. (38).

Proof: The proof of Lemma 1 can be decomposed into three steps. Firstly, the asymptotic stability of the closed-loop system described by Eqs. (34) and (36) has to be proved and ensured under the developed control action. Secondly, it has to be shown that the states variables describing the nonlinear model Eq. (5) should converge to zero as $t \to \infty$. The last step of proof is to show and ensure that the globally asymptotic convergence to zero.

The first step is verified by suggesting a Lyapunov function candidate given by;

$$V = \frac{1}{2}z_3^2 + \frac{1}{2}z_4^2 + \frac{1}{2}\lambda_2\chi_1^2 \tag{44}$$

Differentiating both sides of Eq. (44) along with the solutions of the system described by Eqs. (32), (34) and (36) which results in:

$$\dot{V} = -c_3 z_3^2 - c_4 z_4^2 \le 0 \tag{45}$$

It is clear from the above equation that the inequality $V(t) \le V(0)$ is verified and, hence, the states χ_1 , z_3 and z_4 are bounded and consequently $\dot{z}_3, \dot{z}_4$ are also bounded. The second derivative of a Lyapunov function can easily be computed as:

$$\ddot{V} = -2c_3 z_3 \dot{z}_3 - 2c_4 z_4 \dot{z}_4 \tag{46}$$

Since z_3 , z_4 , $\dot{z}_3$ and $\dot{z}_4$ are all bounded, therefore, $\ddot{V}$ is also bounded. Barbalat's Lemma can be applied to show that both z_3 and z_4 converge to zero as $t \to \infty$. Since the zero convergence of z_3 has been already confirmed and the parameters k_3 and k_4 are merely constants, then The GAS of the zero dynamic in Eq. (43) indicates that the q_1 and p_1 will asymptotically converge to desired reference trajectory. Since q_1 and p_1 are orthogonal to each other, additionally, from Eqs. (28) and (29) as $t \to \infty$ indicates:

$$\lim_{t\to\infty} e = \lim_{t\to\infty}\left[q_1 - q_{1d}\right] = 0 \tag{47}$$

$$\lim_{t\to\infty} \dot{e} = \lim_{t\to\infty}\left[p_1 - \dot{q}_{1d}\right] = 0 \tag{48}$$

From Eqs. (47) and (48) the convergence of $e,\dot{e}$ to zero as $t\to\infty$ has been proved, from Eq. (27) the following can be concluded;

$$\lim_{t\to\infty} q_2 = \lim_{t\to\infty}\left[-k_3 e - k_4 m_{12}\cos q_2 \dot{e} - k_4 m_{22} p_2\right] = 0 \tag{49}$$

Equations (31) and (33), lead to the following reasoning;

$$\lim_{t\to\infty} p_2 = \lim_{t\to\infty} \alpha = 0 \tag{50}$$

Therefore, the convergence of e and $\dot{e}$ to zero as $t\to\infty$ this leads to the fact

$$q_2 + k_4 m_{22} p_2 \to 0 \tag{51}$$

The above Eq. (51) must converge to zero when z_3 converges to the zero. Since q_2 and p_2 are orthogonal to each other. Then the individual element q_2 and p_2 must converge to zero as $t\to\infty$. Hence, the proposed control law guarantees the global stabilization of the ball and arc system.

7. Simulation Results

In this section, the developed block-backstepping algorithms are implemented, for both linearized and nonlinear system, within the environment of MATLAB software. The MATLAB code is developed inside an M-file and 4th order Runge-Kutta are used for the numerical solution. It has been shown that 0.01 second sampling time is appropriate to guarantee the stability of the numerical solution and to give suitable plot resolution. The appropriate and model of ball and arc system using MATLAB package. The numerical physical parameters of the system are listed in Table 1.

Table 1. Physical parameters of ball and arc system [5].

System parameter	Value
Mass of the cart and arc (*M*)	2
Mass of the ball (*m*)	0.05
Gravitational acceleration (*g*)	9.81
Moment of inertia of the ball (*I*)	2.88×10^{-6}
Radius of the ball (*r*)	0.012
Radius of the arc (*R*)	0.08
Motor constant (k_m)	0.0534
Radius of the pinion (R_1)	0.08
Motor armature resistance (R_a)	1.6979

Also, the numerical values of design parameters used through the design of block-backstepping control algorithms are chosen as given in Table 2. For the linearized system c_1 and c_2 are selected based on Eq. (23) to make $\dot{V}$ negative definite. Additionally, k_1 and k_2 indicated in Eq. (26) has been chosen to guarantee the GAS of zero dynamic in Eq. (25). λ is a positive design constant. For non-linear system c_3 and c_4 are selected according to Eq. (45) to make $\dot{V}$ negative definite. The value of k_3 and k_4 in Table 2 make the zero dynamics in Eq. (43) to behave as a stable focus as indicated in Fig. 3.

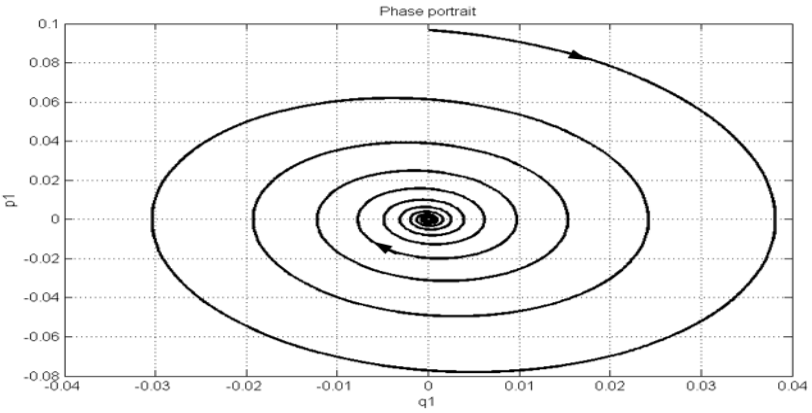


Fig. 3. The phase portrait.

Table 2. Design constant of block backstepping controller.

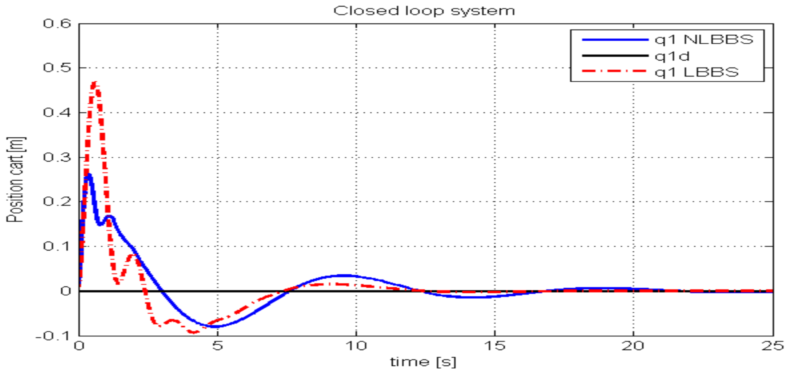
Design constants	Value	Design constants	Value
k_1	0.05	k_3	0.05
k_2	15	k_4	8
c_1	2	c_3	5
c_2	90	c_4	35
λ_1	20	λ_2	60

Firstly, the initial conditions used to start the simulation for both non-linear and linearized systems, based on their associated block backstepping controllers, are set to $[q_1(0) \;\; p_1(0) \;\; q_2(0) \;\; p_2(0)]^T = [0 \;\; 0 \;\; 0.523 \;\; 0]^T$.

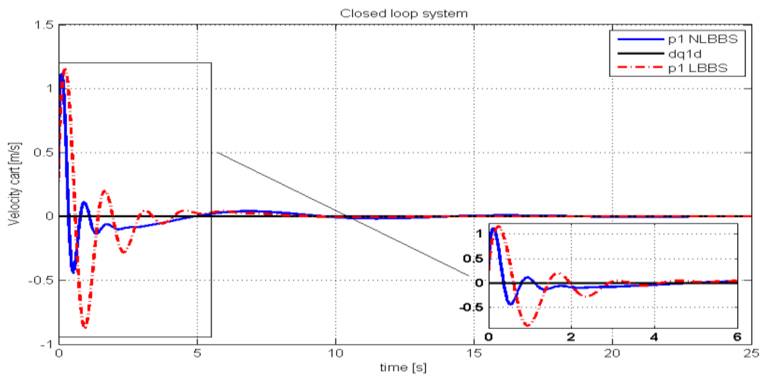
Figures 4(a) and 4(b) show that cart position and velocity responses reach the steady state in 12 and 5 seconds for both systems, respectively. However, the angular and velocity of ball reach the equilibrium point in 1.5 and 3 seconds for the nonlinear and linear controller, respectively, as indicated in Figs. 4(c) and 4(d). The controller actuating signals is shown in Fig. 4(e). The behaviours of force action for both controlled systems are illustrated in Fig. 4(f).

The tracking performance of both controllers for their associated systems and for the above initials are depicted in Fig. 5. The figure shows that both controllers perform well for this particular initial states such that they could stabilize the ball angular position to zero angle location. However, the nonlinear controller shows better transient characteristics than the linear one. To observe the performance of both controllers for a larger initial deviation of the ball, the initial condition of states are set to the following initial state vector:

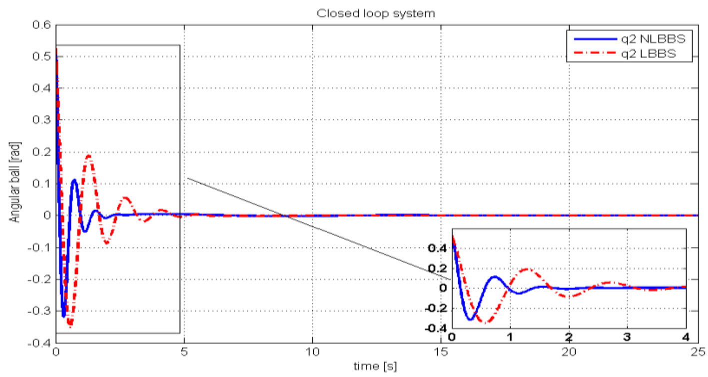
$$[q_1(0) \;\; p_1(0) \;\; q_2(0) \;\; p_2(0)]^T = [0 \;\; 0 \;\; 1.22 \;\; 0]^T$$



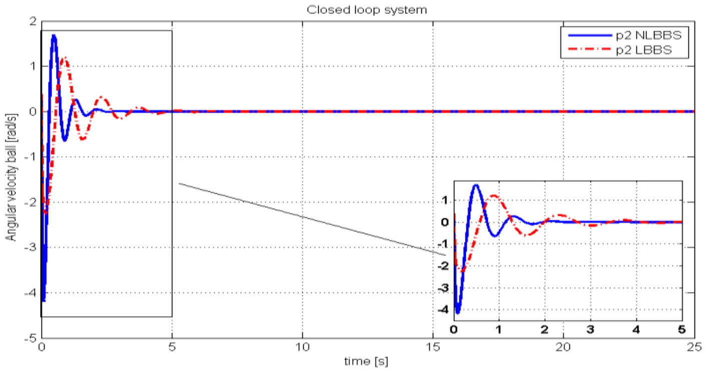
(a) Cart displacement.



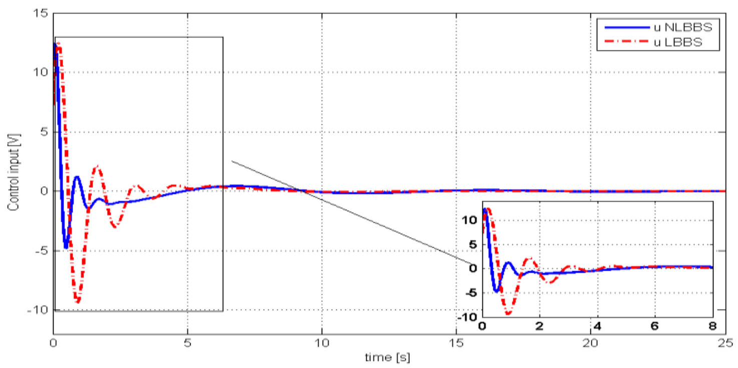
(b) Cart linear velocity.



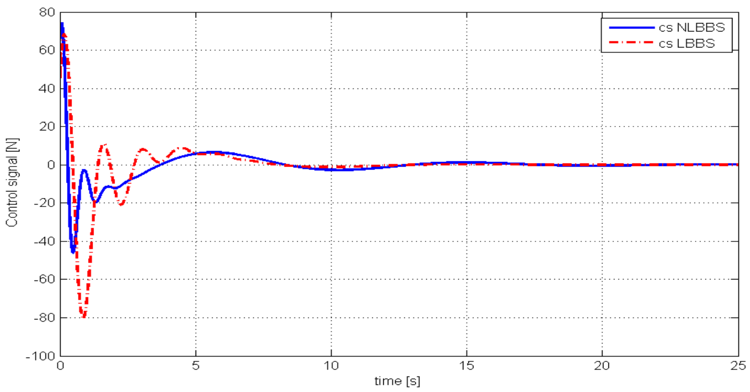
(c) Ball angular position.



(d) Ball angular velocity.

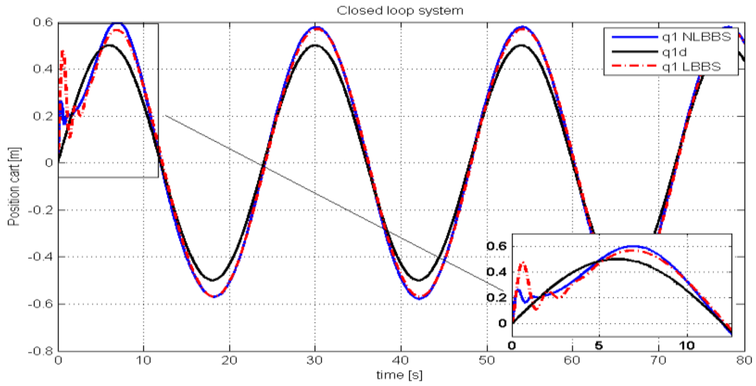


(e) Control input.

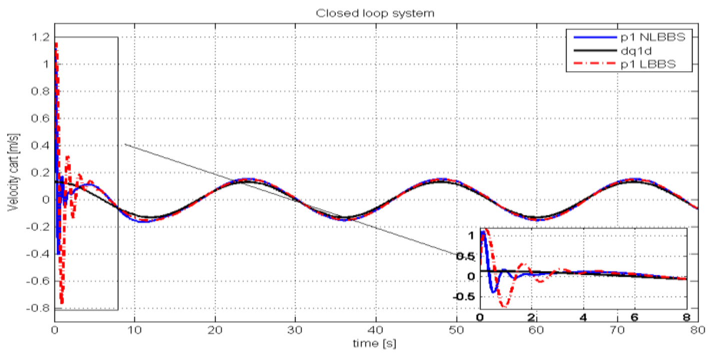


(f) Behaviours of force actions.

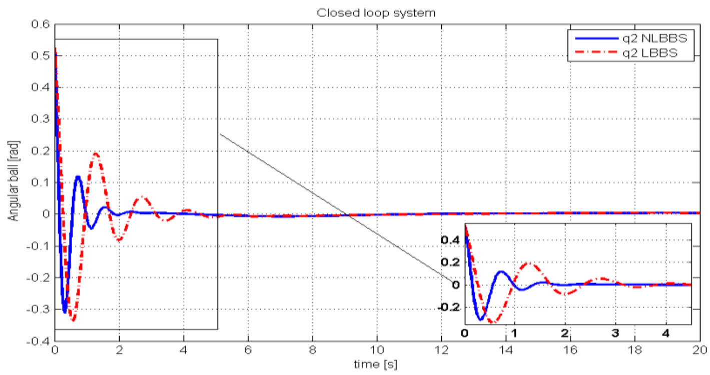
Fig. 4. Responses of the system for a small initial deviation of the ball.



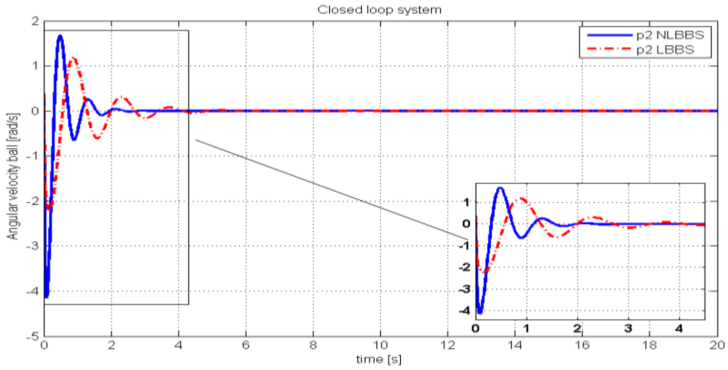
(a) Cart displacement.



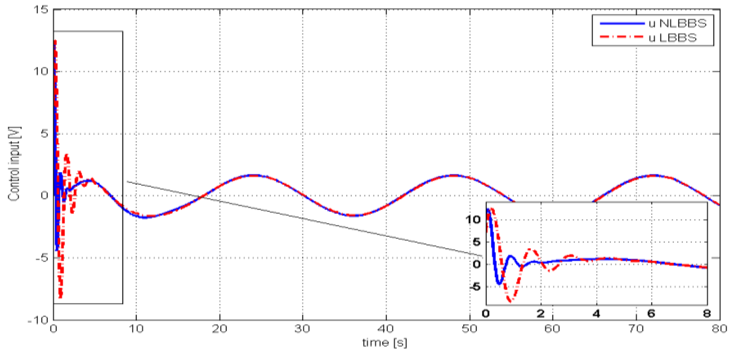
(b) Cart linear velocity.



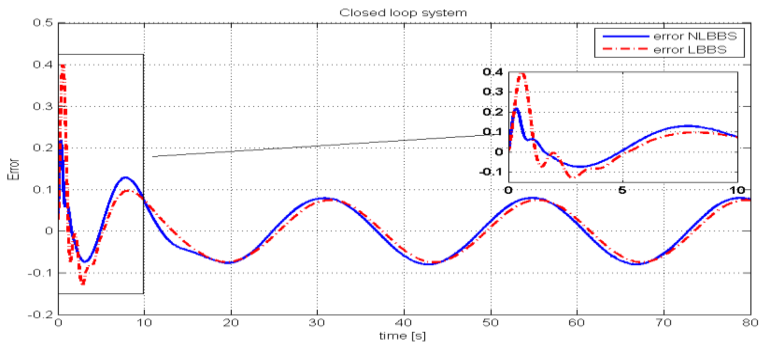
(c) Ball angular position.



(d) Ball angular velocity.



(e) Control input.



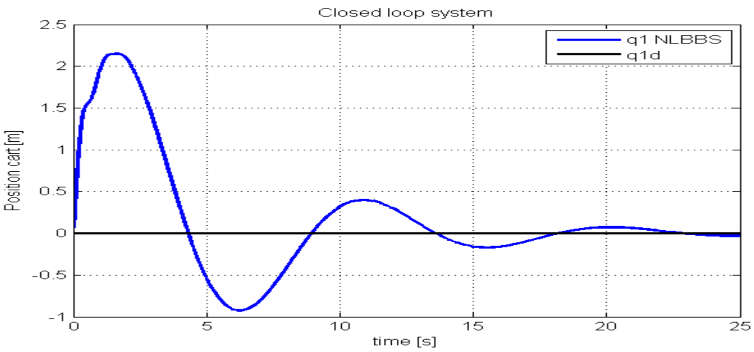
(f) Error between actual and desired.

Fig. 5. Tracking performance with a small initial deviation of the ball.

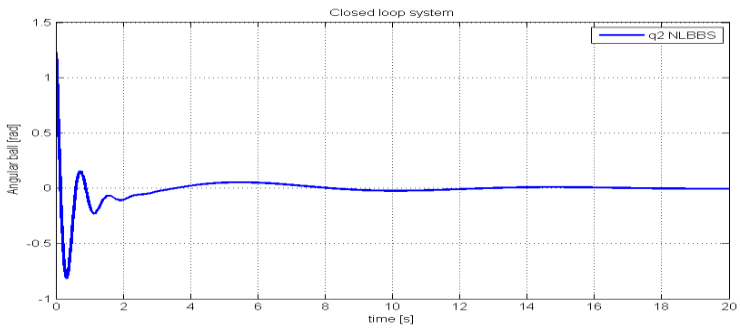
It is clear that the initial state $q_2(0)$ is far away from the equilibrium state. The block-backstepping controller designed for the linearized system has failed to stabilize the system and even worse its relevant responses increased without binding. On the other side, the block-backstepping controller could successfully bring the states to equilibrium point and guarantee the system stability as indicated in Figs. 6(a) and 6(b).

The effectiveness of integral action on the robust characteristics, dynamic behaviour and steady-state error have been assessed for the only backstepping controller of the nonlinear system. Figure 7 has investigated the cases of including and excluding the integral action to the block-backstepping controller of a non-linear system for the first initial condition $q_2(0)$ =0.523. It is clear that the addition of integral action could confine the excursions of cart displacement and ball angular position to lower levels.

In Fig. 8, a disturbance pulse of height 0.5 N is exerted to the system during the period (20-20.1) seconds. The effect of applied disturbance on the controlled system has been shown in Fig. 9. It is clear from this figure that the presence of integral action could enhance the robustness of the nonlinear-controlled system in the presence of parameter variation (disturbance). The effect of the integral action on steady-state characteristics has been evaluated by calculating the steady-state error at the end of runtime. In the presence of integral action, it is found that the steady state error of the cart displacement is equal to 2.53 mm, while that for ball angular position is equal to 0.000617 rad. On the other hand, in the absence of integral action and using the same parameters of Table 2, it has been shown that the steady state for cart displacement is equal to 17.49 mm and that for ball angular position is equal to 0.00301 rad.

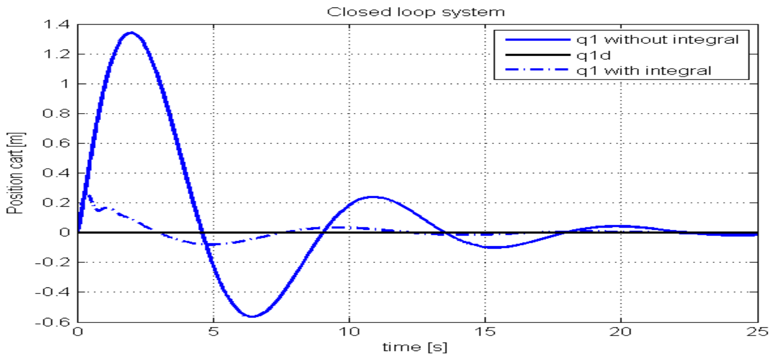


(a) Cart displacement.

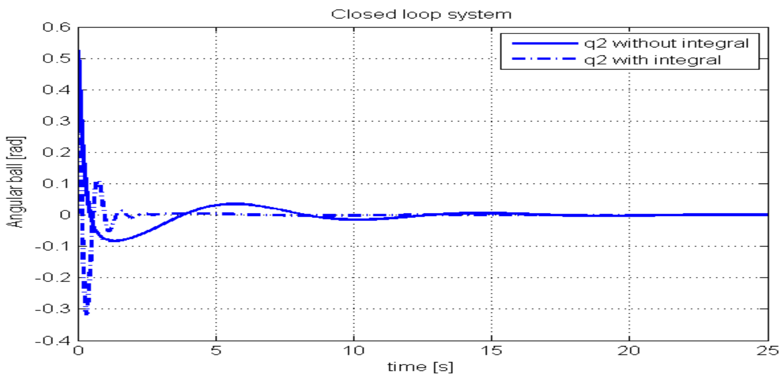


(b) Ball angular position.

Fig. 6. The performance of block-backstepping controller for nonlinear system with a large initial ball deviation.



(a) Cart displacement.



(b) Ball angular position.

Fig. 7. The performance of block-backstepping controller for nonlinear system with and without integral action.

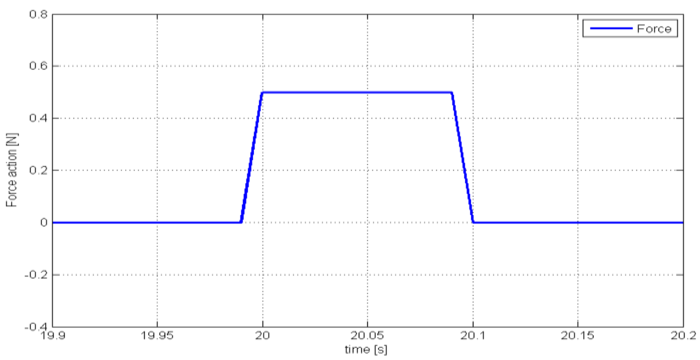
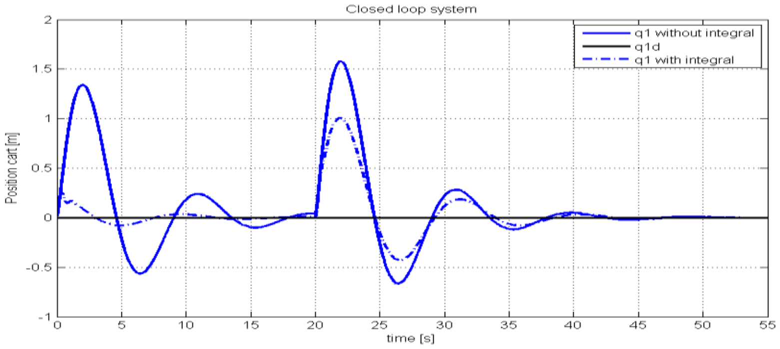
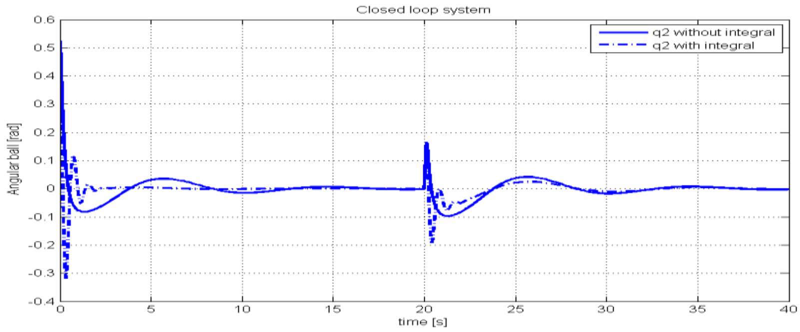


Fig. 8. Force action during the period (20, 20.1).



(a) Cart displacement.



(b) Ball angular position.

Fig. 9. The robustness examination of the block-backstepping controller for the non-linear system under disturbance with and without integral control action.

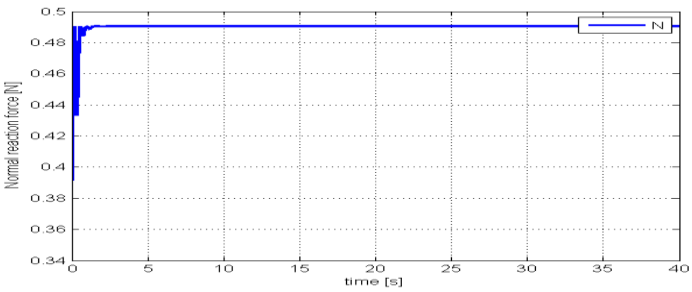


Fig. 10. Behavior of normal reaction force.

It is interesting to show the behaviour of the normal reaction force (N) satisfying the condition of Eq. (4). This force dynamic is depicted in Fig. 10.

8. Conclusions

In this work, the design of block backstepping algorithm is developed for both nonlinear and linearized versions of ball and arc system. The simulated results showed that in spite that both structures of designed controllers perform well for solving both regulation and tracking problem, the block backstepping designed for the considered nonlinear system can cope with a larger excursion of the ball and can bring the desired state to equilibrium point in asymptotically stable manner. On the other hand, the block-backstepping controller based on integral action could enhance both steady-state characteristics and closed system robustness.

Nomenclatures

c_i	Positive design constants for the controller
F	The force applied to the cart, N
g	Gravitational acceleration, m/s^2
I	Moment of inertia of the ball, kg/m^2
k_i	Design constants for the controller
k_m	Motor constant, N m/A
M	Mass of the cart and arc, kg
m	Mass of the ball, kg
O_a	The center of the arc
O_b	The center of the ball
p_1	The velocity of the cart, m/s
p_2	The angular velocity of the ball, rad/s
q_1	The displacement of the cart, m
q_2	The angular displacement, red
R	Radius of the arc, m
R_a	Motor armature resistance, ohm
R_1	Radius of the pinion, m
r	Radius of the ball, m
u	The control input of the ball and arc system, V

Greek Symbols

λ_ι	Arbitrary positive design constant
α	Stabilizing function

Abbreviations

DOF	Degree of Freedom
GAS	Global Asymptotic Stability
MIMO	Multiple Input Multiple Output
T-S	Takagi Sugeno
UMSs	Underactuated Mechanical Systems

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